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Annihilation poles of a Smirnov-type integral formula for solutions to the quantum Knizhnik–Zamolodchikov equation

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Abstract. We consider the recently obtained integral representation of the quantum Knizhnik–Zamolodchikov equation of level 0. We obtain the condition for the integral kernel such that these solutions satisfy three axioms for the form factor in the manner of Smirnov. We discuss the relation between this integral representation and the form factor of the XXZ spin chain.

1. Introduction

In [1] an integral formula of the Smirnov-type was given for the quantum Knizhnik–Zamolodchikov (q-KZ) equation [2] of level 0 associated with the vector representation of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}_2})$. The $U_q(\widehat{\mathfrak{sl}_n})$ generalization was studied in [3]. In these formulae, the freedom of solutions to the q-KZ equation corresponds to the choice of integral kernel with the cycle of integration being fixed. This paper is a step towards the determination of the integral kernel given in [1] by studying the annihilation pole structures of the solutions.

In a pioneering work [4], Smirnov constructed the integral formulae for the form factors of the sine–Gordon model that satisfy three axioms: (i) S-matrix symmetry; (ii) (deformed) cyclicity; (iii) annihilation pole condition. He utilized these axioms to construct the matrix elements of local operators. References [1, 3] were based on Smirnov's observation [5] that (i) and (ii) imply the q-KZ equation of level 0. In these works, instead of solving the q-KZ equation directly, a system of difference equations arising from (i′) the R-matrix symmetry and (ii) the deformed cyclicity were considered. At this moment the integral kernel of the formula is arbitrary except that it satisfies appropriate symmetries and quasi-periodicity conditions. These results can be easily modified so as to satisfy (i) instead of (i′) for the S-matrix having crossing symmetry. In this paper we shall derive the condition for the integral kernel such that these solutions satisfy the third axiom for the $U_q(\widehat{\mathfrak{sl}_2})$ case.

The q-KZ equation was originally introduced by Frenkel and Reshetikhin [2] and was found to be the master equation for the form factors of solvable lattice models with quantum affine symmetries [6, 7]. In this approach, the form factors can be calculated by utilizing the vertex operators of the algebra. (See [8, 9] for explicit calculations.) Moreover, the form

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factors of appropriate operators were shown to satisfy Smirnov's three axioms [7, 10, 11]. Since our S-matrix coincides with the one appearing in the XXZ model, our solutions are expected to be related to the form factors of some operators in the model.

This paper is organized as follows. In section 2 we formulate the problem and summarize our result. In sections 3 and 4 we prove our result. In section 5 we discuss our solutions in the context of the XXZ model.

2. Problem and result

The purpose of this section is to formulate the problem, thereby fixing our notations, and to state our result.

For a fixed complex parameter q such that $0 < |q| < 1$, let $U = U'_q(\widehat{\mathfrak{sl}_2})$ be a $\mathbb{C}$ algebra generated by e_i, f_i and t_i, $i = 0, 1$, that satisfy

$$[e_i, f_j] = \delta_{ij}\frac{t_i - t_i^{-1}}{q - q^{-1}} \qquad t_i e_j t_i^{-1} = q^{4\delta_{ij}-2} e_j \qquad t_i f_j t_i^{-1} = q^{2-4\delta_{ij}} f_j$$
$$t_i t_j = t_j t_i \qquad t_i t_i^{-1} = t_i^{-1} t_i = 1$$

and the Serre relations [12]. Let Δ be the following coproduct of U:

$$\Delta(e_i) = e_i \otimes 1 + t_i \otimes e_i \qquad \Delta(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i \qquad \Delta(t_i) = t_i \otimes t_i$$

and set $\Delta' = \sigma \circ \Delta$, where $\sigma(x \otimes y) = y \otimes x$ for $x, y \in U$. Set $V \cong \mathbb{C}v_+ \oplus \mathbb{C}v_-$ and let (π_ζ, V), $\zeta \in \mathbb{C}\backslash\{0\}$, signify the vector representation of U defined by

$$\begin{aligned}
&\pi_\zeta(e_1)(v_+, v_-) = \zeta(0, v_+) \qquad \pi_\zeta(f_1)(v_+, v_-) = \zeta^{-1}(v_-, 0)\\
&\pi_\zeta(t_1)(v_+, v_-) = (qv_+, q^{-1}v_-) \qquad \pi_\zeta(e_0)(v_+, v_-) = \zeta(v_-, 0)\\
&\pi_\zeta(f_0)(v_+, v_-) = \zeta^{-1}(0, v_+) \qquad \pi_\zeta(t_0)(v_+, v_-) = (q^{-1}v_+, qv_-).
\end{aligned} \tag{2.1}$$

Later we shall use the following abbreviation for its tensor product representation via Δ:

$$\pi_{(\zeta_1,\ldots,\zeta_N)}(y) = (\pi_{\zeta_1} \otimes \cdots \otimes \pi_{\zeta_N}) \circ \Delta^{(N-1)}(y) \qquad y \in U.$$

Let $R(\zeta) \in \mathrm{End}(V \otimes V)$ be the R-matrix of the six vertex model

$$R(\zeta) v_{\varepsilon'_1} \otimes v_{\varepsilon'_2} = \sum_{\varepsilon_1,\varepsilon_2} v_{\varepsilon_1} \otimes v_{\varepsilon_2} R(\zeta)^{\varepsilon_1\varepsilon_2}_{\varepsilon'_1\varepsilon'_2}$$

where the non-zero entries are given by

$$\begin{aligned}
&R(\zeta)^{++}_{++} = R(\zeta)^{--}_{--} = 1\\
&R(\zeta)^{+-}_{+-} = R(\zeta)^{-+}_{-+} = b(\zeta) = \frac{(1-\zeta^2)q}{1-\zeta^2q^2}\\
&R(\zeta)^{+-}_{-+} = R(\zeta)^{-+}_{+-} = c(\zeta) = \frac{(1-q^2)\zeta}{1-\zeta^2q^2}.
\end{aligned}$$

Then the following intertwining property holds [12]:

$$R(\zeta_1/\zeta_2)(\pi_{\zeta_1} \otimes \pi_{\zeta_2}) \circ \Delta(y) = (\pi_{\zeta_1} \otimes \pi_{\zeta_2}) \circ \Delta'(y) R(\zeta_1/\zeta_2). \tag{2.2}$$

We introduce the scattering matrix S [6, 7]:

$$S(\zeta) = S_0(\zeta) R(\zeta)$$

where

$$S_0(\zeta) = \zeta \frac{(z^{-1};q^4)_\infty (zq^2;q^4)_\infty}{(z;q^4)_\infty (z^{-1}q^2;q^4)_\infty} \qquad (a;p_1,\ldots,p_n)_\infty = \prod_{k_i \geqslant 0} (1 - ap_1^{k_1}\cdots p_n^{k_n})$$

and $z = \zeta^2$. In what follows we shall work with the tensor products of the vector space V. Following the usual convention, for $M \in \mathrm{End}(V)$ we let M_j denote the operator on $V^{\otimes N}$ acting as M on the jth tensor component and as the identity on the other components. Similarly for $X = S$ or R, we let $X_{jk}(\zeta)$ $(j \neq k)$ signify the operator on $V^{\otimes N}$ acting as $X(\zeta)$ on the (j,k)th tensor components and as the identity on the other components. In particular, we have $X_{kj}(\zeta) = P_{jk}X_{jk}(\zeta)P_{jk}$, where $P \in \mathrm{End}(V \otimes V)$ stands for the transposition $P(x \otimes y) = y \otimes x$. We often use the abbreviations

$$X_{1,\ldots,N|N+1}(\zeta_1,\ldots,\zeta_N|\zeta_{N+1}) = X_{1,N+1}(\zeta_1/\zeta_{N+1})\cdots X_{N,N+1}(\zeta_N/\zeta_{N+1})$$
$$X_{N+1|1,\ldots,N}(\zeta_{N+1}|\zeta_1,\ldots,\zeta_N) = X_{N+1,N}(\zeta_{N+1}/\zeta_N)\cdots X_{N+1,1}(\zeta_{N+1}/\zeta_1)$$

where $X = S$ or R.

The main properties of $S(\zeta)$ are the Yang–Baxter equation

$$S_{12}(\zeta_1/\zeta_2)S_{13}(\zeta_1/\zeta_3)S_{23}(\zeta_2/\zeta_3) = S_{23}(\zeta_2/\zeta_3)S_{13}(\zeta_1/\zeta_3)S_{12}(\zeta_1/\zeta_2) \tag{2.3}$$

the initial condition

$$S(1) = -P \tag{2.4}$$

the unitarity relation

$$S_{12}(\zeta_1/\zeta_2)S_{21}(\zeta_2/\zeta_1) = 1 \tag{2.5}$$

and the crossing symmetry

$$S_{12}(\zeta)v_\varepsilon \otimes u_\sigma = \sigma S_{31}(-\sigma/q\zeta)v_\varepsilon \otimes u_\sigma \tag{2.6}$$

where $u_\sigma = v_- \otimes v_+ + \sigma v_+ \otimes v_-$, $\sigma = \pm$.

Let $V^{(nl)}$ be the subspace of $V^{\otimes N}$ defined by

$$V^{(nl)} = \oplus\, \mathbb{C}v_{\varepsilon_1} \otimes \cdots \otimes v_{\varepsilon_N}$$

where the sum is taken over $\varepsilon_j = \pm 1$ with fixed

$$n = \sharp\{j \mid \varepsilon_j = -\} \qquad l = \sharp\{j \mid \varepsilon_j = +\} \qquad (n + l = N)$$

and let us consider a $V^{(nl)}$-valued function

$$G_\varepsilon^{(nl)}(\zeta_1,\ldots,\zeta_N) = \sum v_{\varepsilon_1} \otimes \cdots \otimes v_{\varepsilon_N} G_\varepsilon^{(nl)}(\zeta_1,\ldots,\zeta_N)^{\varepsilon_1\cdots\varepsilon_N} \qquad (\varepsilon = \pm).$$

Our problem is to obtain the function family $G_\varepsilon^{(nl)}(\zeta_1,\ldots,\zeta_N)$, $\varepsilon = \pm$, $n, l = 1, 2, \ldots$, that satisfy the following three axioms.

Axiom 1. S-matrix symmetry

$$P_{j\,j+1}G_\varepsilon^{(nl)}(\ldots,\zeta_{j+1},\zeta_j,\ldots) = S_{j\,j+1}(\zeta_j/\zeta_{j+1})G_\varepsilon^{(nl)}(\ldots,\zeta_j,\zeta_{j+1},\ldots) \qquad (1 \leqslant j \leqslant N-1). \tag{2.7}$$

Axiom 2. Deformed cyclicity

$$P_{12}\cdots P_{N-1N}G_\varepsilon^{(nl)}(\zeta_2,\ldots,\zeta_N,\zeta_1 q^{-2}) = r_\varepsilon^{(l-n)}(\zeta_1)D_1^{(l-n)}G_\varepsilon^{(nl)}(\zeta_1,\ldots,\zeta_N). \tag{2.8}$$

Axiom 3. Annihilation pole condition. The $G_\varepsilon^{(nl)}(\zeta)$ has poles at $\zeta_N = -\sigma\zeta_{N-1}/q$ $(\sigma = \pm)$ and the residue is given by

$$\mathrm{Res}_{\zeta_N/(-\sigma\zeta_{N-1}q^{-1})=1}\, G_\varepsilon^{(nl)}(\zeta) = \frac{1}{2}\left(I - \sigma^{N+1} r_\varepsilon^{(l-n)}(-\sigma\zeta_{N-1}q) D_N^{(l-n)} S_{N-1|1,\ldots,N-2}(\zeta_{N-1}|\zeta')\right) \times G_{\sigma\varepsilon}^{(n-1l-1)}(\zeta') \otimes u_\sigma. \tag{2.9}$$

Here $r_\varepsilon^{(k)}(\zeta) = \varepsilon\, r^{(k)}(\zeta)$, $r^{(k)}(\zeta)$ are scalar functions satisfying

$$r^{(k)}(\zeta) r^{(k)}(-\sigma\zeta q) = \sigma^N \tag{2.10}$$

$D^{(k)} = \mathrm{diag}(\delta^{(k)}, \delta^{(k)-1})$, and I is the identity operator. In this paper we employ the convention $\mathrm{Res}_{x/y=1}\, G(x) = F(y)$ when $G(x) = \frac{1}{x/y-1}F(y) + \mathrm{O}(1)$ at $x \simeq y$, and we often use the abbreviations $(\zeta) = (\zeta_1, \ldots, \zeta_N)$ and $(\zeta') = (\zeta_1, \ldots, \zeta_{N-2})$. In section 5 we shall discuss the physical meaning of the above axioms and the reason why we introduce $r^{(k)}(\zeta)$.

Remark 1. The action of the scattering matrix $S(\zeta)$ preserves the values of n and l and the S-matrix also satisfies $\mathbb{Z}_2$-symmetry. Therefore, we can consider $G_\varepsilon^{(nl)}(\zeta) \in V^{(nl)}$ and may assume $n \leqslant l$ without loss of generality.

Remark 2. The condition for $r^{(k)}(\zeta)$ (2.10) follows from the consistency of the three axioms. This can be seen from (4.3) and the argument below it. Similarly we can show that the diagonal operator $r_\varepsilon^{(k)}(\zeta)D^{(k)}$ depends only on $l - n$.

Hereafter we shall restrict ourselves to the case $\delta^{(k)} = q^{-k/2}$ and $n \leqslant l$, since for this choice the solutions to the first two axioms have been obtained in [1]. In this paper, we shall derive the condition for these solutions to satisfy the third axiom. Before we state our result, we shall explain how the solutions to the first two axioms are obtained from the results of [1]. Set

$$G_\varepsilon^{(nl)}(\zeta) = \overline{G}_\varepsilon^{(nl)}(\zeta) \prod_{1\leqslant i<j\leqslant N} g(z_i/z_j) \prod_{j=1}^{N} \zeta_j^{j-N} \tag{2.11}$$

where

$$g(z) = \frac{(z; q^4, q^4)_\infty (q^4/z; q^4, q^4)_\infty}{(q^2 z; q^4, q^4)_\infty (q^6/z; q^4, q^4)_\infty}.$$

Then, the first two axioms can be recast as

$$P_{j\,j+1} \overline{G}_\varepsilon^{(nl)}(\ldots, \zeta_{j+1}, \zeta_j, \ldots) = R_{j\,j+1}(\zeta_j/\zeta_{j+1}) \overline{G}_\varepsilon^{(nl)}(\ldots, \zeta_j, \zeta_{j+1}, \ldots) \tag{2.12}$$

$$P_{12} \cdots P_{N-1N} \overline{G}_\varepsilon^{(nl)}(\zeta_2, \ldots, \zeta_N, \zeta_1 q^{-2}) = r_\varepsilon(\zeta_1) D_1^{(l-n)} \overline{G}_\varepsilon^{(nl)}(\zeta_1, \ldots, \zeta_N) \prod_{j=2}^{N} \zeta_j/\zeta_1. \tag{2.13}$$

Therefore, though the deformed cyclicity (2.13) is different from that given in [1] by a multiplication factor, $\overline{G}_\varepsilon^{(nl)}(\zeta)$ is similarly shown to have the integral formula

$$\overline{G}_\varepsilon^{(nl)}(\zeta) = \frac{1}{m!} \prod_{\mu=1}^{m} \oint_{C^{(N)}} \frac{\mathrm{d}x_\mu}{2\pi \mathrm{i}} \Psi_\varepsilon^{(nl)}(x_1, \ldots, x_m|\zeta_1, \ldots, \zeta_N) \langle \Delta^{(nl)} \rangle (x_1, \ldots, x_m|\zeta_1, \ldots, \zeta_N). \tag{2.14}$$

Here $m = n - 1$ for $n = l$ and $m = n$ for $n < l$, and $\Delta^{(nl)}(x_1, \ldots, x_m|z_1, \ldots, z_n|z_{n+1}, \ldots, z_N)$ is the same polynomial as obtained in [1]. (See

(3.7) below, and [1] for further details.) $\langle\Delta^{(nl)}\rangle(x_1,\ldots,x_m|\zeta_1,\ldots,\zeta_N)\in V^{(nl)}$ is the vector defined from this polynomial as

$$\langle\Delta^{(nl)}\rangle(x_1,\ldots,x_m|\zeta_1,\ldots,\zeta_N)^{-\cdots-+\cdots+} = \frac{\Delta^{(nl)}(x_1,\ldots,x_m|z_1,\ldots,z_n|z_{n+1},\ldots,z_N)}{\prod_{j=1}^{n}\prod_{i=n+1}^{N}(z_i - z_j\tau^2)}\prod_{j=1}^{n}\zeta_j$$

$$P_{j\,j+1}\langle\Delta^{(nl)}\rangle(x_1,\ldots,x_m|\ldots,\zeta_{j+1},\zeta_j,\ldots)$$
$$= R_{j\,j+1}(\zeta_j/\zeta_{j+1})\langle\Delta^{(nl)}\rangle(x_1,\ldots,x_m|\ldots,\zeta_j,\zeta_{j+1},\ldots)$$

where, and hereafter, $\tau = q^{-1}$. The integration $\oint_{C^{(N)}} dx_\mu$ is along a simple closed curve $C^{(N)} = C^{(N)}(z_1,\ldots,z_N)$ oriented anti-clockwise, which encircles the points $z_j\tau^{-1-4k}$, $1 \leqslant j \leqslant N$, $k \geqslant 0$, but not $z_j\tau^{1+4k}$, $1 \leqslant j \leqslant N$, $k \geqslant 0$. The kernel $\Psi_\varepsilon^{(nl)}$ has the form

$$\Psi_\varepsilon^{(nl)}(x_1,\ldots,x_m|\zeta_1,\ldots,\zeta_N) = \vartheta_\varepsilon^{(nl)}(x_1,\ldots,x_m|\zeta_1,\ldots,\zeta_N)\prod_{\mu=1}^{m}\prod_{j=1}^{N}\psi(x_\mu/z_j).$$

Here

$$\psi(z) = \frac{1}{(zq;q^4)_\infty(z^{-1}q;q^4)_\infty}$$

and $\vartheta_\varepsilon^{(nl)}$ is an arbitrary function that has the following properties:

(i) it is anti-symmetric and holomorphic in the $x_\mu \in \mathbb{C}\backslash\{0\}$;

(ii) it is symmetric and meromorphic in the $\zeta_j \in \mathbb{C}\backslash\{0\}$;

(iii) it has the two transformation properties

$$\frac{\vartheta_\varepsilon^{(nl)}(x_1,\ldots,x_m|\zeta_1,\ldots,\zeta_j\tau^2,\ldots,\zeta_N)}{\vartheta_\varepsilon^{(nl)}(x_1,\ldots,x_m|\zeta_1,\ldots,\zeta_N)} = \tau^{N/2}r_\varepsilon^{(l-n)}(\zeta_j)\prod_{\mu=1}^{m}\frac{-z_j\tau}{x_\mu}\prod_{\substack{k=1\\k\neq j}}^{N}\frac{\zeta_k}{\zeta_j} \tag{2.15}$$

$$\frac{\vartheta_\varepsilon^{(nl)}(x_1,\ldots,x_\mu\tau^4,\ldots,x_m|\zeta_1,\ldots,\zeta_N)}{\vartheta_\varepsilon^{(nl)}(x_1,\ldots,x_m|\zeta_1,\ldots,\zeta_N)} = \prod_{j=1}^{N}\frac{-x_\mu\tau}{z_j}. \tag{2.16}$$

Note that the first transformation function (2.15) takes a form different from that given in [1] because of the modification of (2.13).

Now we are in a position to state the main result of this paper.

Theorem 2.1. The function family $G_\varepsilon^{(nl)}(\zeta)$ defined in (2.11) and (2.14) satisfies the annihilation pole condition (2.9) if $\vartheta_\varepsilon^{(nl)}$ satisfies the recurrence relation

$$\frac{\vartheta_\varepsilon^{(nl)}(x_1,\ldots,x_{m-1},z_{N-1}\tau|\zeta_1,\ldots,\zeta_{N-2},\zeta_{N-1},-\sigma\zeta_{N-1}\tau^{-1})}{\vartheta_{\sigma\varepsilon}^{(n-1,l-1)}(x_1,\ldots,x_{m-1}|\zeta_1,\ldots,\zeta_{N-2})}$$
$$= c^{(nl)}z_{N-1}^{N-m-1}\prod_{\mu=1}^{m-1}\theta(qx_\mu/z_{N-1}|q^2). \tag{2.17}$$

where

$$\theta(z|q) = (z;q)_\infty(q/z;q)_\infty$$
$$c^{(nl)} = (-\tau)^{m-n-2l+3}(q^2;q^2)_\infty^2\frac{(q^4;q^4,q^4)_\infty^2}{(q^2;q^4,q^4)_\infty^2}.$$

There exists a non-trivial example of the function $\vartheta_\varepsilon^{(nl)}$ which satisfies the above conditions for a particular choice of $r^{(k)}(\zeta)$. (See section 5.) We shall check the $(-\cdots-+\cdots+)$ component of the annihilation pole condition (2.9) in the next section and show that theorem 2.1 follows from it in section 4.

3. The annihilation pole condition for the extreme component

In this section we consider the $(-\cdots-+\cdots+)$ component of the annihilation pole condition (2.9). In terms of $\overline{G}_\varepsilon^{(nl)}$, it can be recast as the following proposition.

Proposition 3.1. If $\vartheta_\varepsilon^{(nl)}$ satisfies (2.17), then

$$-2\frac{(q^2;q^4,q^4)^2_\infty}{(q^4;q^4,q^4)^2_\infty}\frac{\sigma^{N+1}\tau^{(n-l)/2}\mathrm{Res}_{\zeta_N/(-\sigma\zeta_{N-1}\tau)=1}\overline{G}_\varepsilon^{(nl)}(\zeta)^{-\cdots-+\cdots+}}{r_\varepsilon^{(l-n)}(-\sigma\zeta_{N-1}q)\zeta_{N-1}\prod_{j=1}^{N-2}(\zeta_j\zeta_{N-1}\psi(\tau z_{N-1}/z_j))}$$
$$= \left(R_{N-1|1,\cdots,N-2}(\zeta_{N-1}|\zeta')\overline{G}_{\sigma\varepsilon}^{(n-1l-1)}\otimes u_\sigma\right)^{-\cdots-+\cdots+}. \tag{3.1}$$

Proof. From the definition of the R-matrix, the RHS is equal to

$$\sum_{k=1}^{n} c\left(\frac{\zeta_{N-1}}{\zeta_k}\right)\prod_{j=k+1}^{n} b\left(\frac{\zeta_{N-1}}{\zeta_j}\right)\overline{G}_\varepsilon^{(n-1,l-1)}(\zeta')^{-\cdots\overset{k}{+}\cdots-\overset{n+1}{\mp}\cdots+}. \tag{3.2}$$

Due to the R symmetry (2.12) we have the relation

$$\overline{G}_\varepsilon^{(n-1,l-1)}(\zeta')^{-\cdots\overset{k}{+}\cdots-\overset{n+1}{\mp}\cdots+}$$
$$= \prod_{j=k+1}^{n}\frac{1}{b}\left(\frac{\zeta_k}{\zeta_j}\right)\overline{G}_{\varepsilon,k}^{(n-1l-1)}(\zeta') - \sum_{j=k+1}^{n}\frac{c}{b}\left(\frac{\zeta_k}{\zeta_j}\right)\prod_{\substack{i=k+1\\ i\neq j}}^{n}\frac{1}{b}\left(\frac{\zeta_j}{\zeta_i}\right)\overline{G}_{\varepsilon,j}^{(n-1l-1)}(\zeta')$$

where

$$\overline{G}_{\varepsilon,j}^{(n-1,l-1)}(\zeta') = \overline{G}_\varepsilon^{(n-1l-1)}(\zeta_1,\overset{j}{\hat{\ldots}},\zeta_n,\zeta_j,\zeta_{n+1},\ldots,\zeta_{N-2})^{-\cdots-+\cdots+} \qquad (1\leqslant j\leqslant n).$$

Note that (3.2) contains only one term proportional to $\overline{G}_{\varepsilon,1}^{(n-1l-1)}(\zeta')$ when expressed in terms of $\overline{G}_{\varepsilon,j}^{(n-1l-1)}(\zeta')$. Since the result should be symmetric with respect to $\zeta_1,\ldots,\zeta_n$, we obtain

$$\mathrm{RHS} = \frac{(1-\tau^2)\zeta_{N-1}}{\prod_{j=1}^{n}(z_{N-1}-z_j\tau^2)}\sum_{k=1}^{n}\zeta_k\prod_{\substack{j=1\\ j\neq k}}^{n}\frac{(z_{N-1}-z_j)(z_k-z_j\tau^2)}{z_k-z_j}\overline{G}_{\varepsilon,k}^{(n-1,l-1)}(\zeta'). \tag{3.3}$$

Let us turn to the LHS. In the calculation of the residue of

$$\overline{G}_\varepsilon^{(nl)}(\zeta)^{-\cdots-+\cdots+}$$
$$= \frac{1}{m!}\prod_{\mu=1}^{m}\oint_{C^{(N)}}\frac{\mathrm{d}x_\mu}{2\pi\mathrm{i}}\Psi_\varepsilon^{(nl)}(x_1,\ldots,x_m|\zeta)$$
$$\times\frac{\Delta^{(nl)}(x_1,\ldots,x_m|z_1,\ldots,z_n|z_{n+1},\ldots,z_N)}{\prod_{j=1}^{n}\prod_{i=n+1}^{N}(z_i-z_j\tau^2)}\prod_{j=1}^{n}\zeta_j$$

at $\zeta_N=-\sigma\zeta_{N-1}\tau$, we rewrite the integration as

$$\prod_{\mu=1}^{m}\oint_{C^{(N)}}\frac{\mathrm{d}x_\mu}{2\pi\mathrm{i}} = \prod_{\mu=1}^{m-1}\oint_{C'^{(N)}}\frac{\mathrm{d}x_\mu}{2\pi\mathrm{i}}\left(\oint_{C'^{(N)}}\frac{\mathrm{d}x_m}{2\pi\mathrm{i}} + m\,\mathrm{Res}_{x_m=z_N\tau^{-1}}\right) \tag{3.4}$$

in order to avoid the pinch of the contour $C^{(N)}$. Here $C'^{(N)}$ is a simple anti-clockwise closed curve which encloses the same poles but $z_N\tau^{-1}$ as for $C^{(N)}$, and we have used the symmetry of the integrand with respect to the x_μ's. Since the integrand is regular at $\zeta_N = -\sigma\zeta_{N-1}\tau$, only the second term of the RHS of (3.4) contributes to the residue. Note that from (2.15), (2.17) is equivalent to

$$\frac{\vartheta_\varepsilon^{(nl)}(x_1,\ldots,x_{m-1},z_{N-1}\tau|\zeta',\zeta_{N-1},-\sigma\zeta_{N-1}\tau)}{\vartheta_{\sigma\varepsilon}^{(n-1,l-1)}(x_1,\ldots,x_{m-1}|\zeta')}$$
$$= (-1)^{N-m+1}\tau^{3N/2-m-2}c^{(nl)}\sigma^{N+1}r_\varepsilon^{(l-n)}(-\sigma\zeta_{N-1}q)\prod_{j=1}^{N-2}\zeta_j\zeta_{N-1}$$
$$\times\prod_{\mu=1}^{m-1}\frac{\theta(qx_\mu/z_{N-1}|q^2)}{x_\mu}.$$

We thus find

$$\text{LHS} = \frac{1}{(m-1)!}\prod_{\mu=1}^{m-1}\oint_{C^{(N)}}\frac{dx_\mu}{2\pi i}\Psi_{\sigma\varepsilon}^{(n-1l-1)}(x_1,\ldots,x_{m-1}|\zeta')$$
$$\times\prod_{j=1}^{n}\prod_{i=n+1}^{N-1}\frac{1}{z_i-z_j\tau^2}\prod_{j=1}^{n}\zeta_j(-1)^l\tau^{2-N}\zeta_{N-1}$$
$$\times\frac{\Delta^{(nl)}(x_1,\ldots,x_{m-1},z_{N-1}\tau|z_1,\ldots,z_n|z_{n+1},\ldots,z_{N-1},z_{N-1}\tau^2)}{\prod_{j=1}^{n}(z_{N-1}-z_j)\prod_{\mu=1}^{m-1}(x_\mu-z_{N-1}\tau)}. \tag{3.5}$$

In order to show the equality of (3.3) and (3.5) we shall prove the following proposition.

Proposition 3.2.

$$\frac{\Delta^{(nl)}(x_1,\ldots,x_{m-1},w\tau|z_1,\ldots,z_n|z_{n+1},\ldots,z_{N-2},w,w\tau^2)}{\prod_{j=1}^{n}(w-z_j)\prod_{\mu=1}^{m-1}(x_\mu-w\tau)}$$
$$= \tau^n\sum_{k=1}^{n}\prod_{\substack{j=1\\ j\neq k}}^{n}\frac{w-z_j}{z_k-z_j}\left\{(-\tau)^{l-2}(1-\tau^2)\prod_{i=n+1}^{N-2}(z_i-z_k\tau^2)\Delta^{(n-1l-1)}\right.$$
$$\times(x_1,\ldots,x_{m-1}|z_1,\overset{k}{\hat{\ldots}},z_n|z_k,z_{n+1},\cdots,z_{N-2})$$
$$+\sum_{\nu=1}^{m-1}(-1)^{m+\nu}h^{(N-2)}(x_\nu|z_1,\ldots,z_{N-2})\Delta'^{(m-2)}$$
$$\left.\times(x_1,\overset{\nu}{\hat{\ldots}},x_{m-1}|z_k|z_1,\overset{k}{\hat{\ldots}},z_n|z_{n+1},\ldots,z_{N-2})\right\} \tag{3.6}$$

where $h^{(N)}(x|z_1,\ldots,z_N)$ is the polynomial defined in [1], and

$$\Delta'^{(m-2)}(x_1,\ldots,x_{m-2}|z_1|z_2,\ldots,z_n|z_{n+1},\ldots,z_{N-2})$$
$$= \det\left(\left(A_\lambda^{(n-1l-1)}(x_\mu|z'|z'')\right)_{\substack{1\leqslant\mu\leqslant m-2\\ n-m+1\leqslant\lambda\leqslant n-1}},\left(f_\lambda^{(n-1l-1)}(z_1\tau|z'|z'')\right)_{n-m+1\leqslant\lambda\leqslant n-1}\right)$$

with $z' = (z_2,\ldots,z_n)$ and $z'' = (z_{n+1},\ldots,z_{N-2},z_1)$.

Proof. Let us recall [1] that

$$\Delta^{(nl)}(x_1,\ldots,x_m|z_1,\ldots,z_n|z_{n+1},\ldots,z_N)$$
$$= \det\left(A_\lambda^{(nl)}(x_\mu|z_1,\ldots,z_n|z_{n+1},\ldots,z_N)\right)_{\substack{1\leqslant\mu\leqslant m\\ n-m+1\leqslant\lambda\leqslant n}} \tag{3.7}$$

where

$$
\begin{aligned}
A_\lambda^{(nl)}&(x|a_1,\ldots,a_n|b_1,\ldots,b_l)\\
&=\prod_{j=1}^{n}(x-a_j\tau)f_\lambda^{(nl)}(x|a_1,\ldots,a_n|b_1,\ldots,b_l)\\
&\quad+\tau^{2(l-n+\lambda-1)}\prod_{i=1}^{l}(x-b_i\tau^{-1})g_\lambda^{(n)}(x|a_1,\ldots,a_n).
\end{aligned}
$$

For the definition of $f_\lambda^{(nl)}$ and $g_\lambda^{(n)}$, see [1]. From

$$
\begin{aligned}
A_\lambda^{(nl)}&(w\tau|z_1,\ldots,z_n|z_{n+1},\ldots,z_{N-2},w,w\tau^2)\\
&=\tau^n\prod_{j=1}^{n}(w-z_j)f_\lambda^{(nl)}(w\tau|z_1,\ldots,z_n|z_{n+1},\ldots,z_{N-2},w,w\tau^2)
\end{aligned}
$$

and the determinant structure of $\Delta^{(nl)}$, we find that the LHS of (3.6) is a polynomial in w. From the following properties ((4.3) and (4.15) in [1]),

$$
\begin{gathered}
A_\lambda^{(nl)}(x|a_1,\ldots,a_n|b_1,\ldots,b_l) \text{ is linear with respect to the } b_i\text{'s}\\
f_\lambda^{(nl)}(w\tau|z_1,\ldots,z_n|z_{n+1},\ldots,z_{N-2},w,w\tau^2) \text{ is independent of } w
\end{gathered}
$$

we further find that its degree is equal to $m-1$. Thus, the LHS can be determined from $n(>m-1)$ values at $w=z_k$, $1\leqslant k\leqslant n$. For example, when $w=z_1$, using the recurrence relations of $A_\lambda^{(nl)}$ and $f_\lambda^{(nl)}$ [1], its value is found to be

$$
\tau^n\det\left(\left(A_\lambda^{(n-1l-1)}(x_\mu|z'|z'')\right)_{\substack{1\leqslant\mu\leqslant m-1\\ n-m+1\leqslant\lambda\leqslant n}},\left(f_\lambda^{(n-1l-1)}(z_1\tau|z'|z'')\right)_{n-m+1\leqslant\lambda\leqslant n}\right)
$$

where z' and z'' are the same abbreviations as used in the definition of $\Delta'^{(m-2)}$. Then, noting

$$
f_n^{(n-1l-1)}(z_1\tau|z'|z'')=(-\tau)^{l-2}(1-\tau^2)\prod_{i=n+1}^{N-2}(z_i-z_1\tau^2)
$$

we obtain (3.6). □

After the substitution of (3.6) into (3.5) we can replace $C'^{(N)}(z_1,\ldots,z_N)$ by $C^{(N-2)}(z_1,\ldots,z_{N-2})$. Moreover, due to the same argument as that given in the proof of lemma 3.2 in [1], the terms proportional to $h^{(N-2)}(x_\nu|z_1,\ldots,z_{N-2})$ vanish after the integration. Hence, proposition 3.1 is proved. □

4. Proof of the theorem

In this section we shall complete the proof of theorem 2.1. Firstly, consider the following.

Proposition 4.1. If the function family $G_\varepsilon^{(nl)}(\zeta_1,\ldots,\zeta_N)\in V^{(nl)}$, $\varepsilon=\pm$, $n,l=1,2,\ldots$, satisfies:

(i) S-matrix symmetry (2.7);

(ii) deformed cyclicity (2.8);

(iii) the $(-\cdots-+\cdots+)$ component of the annihilation pole condition (2.9);

(iv) $\pi_{(\zeta_1,\ldots,\zeta_N)}(f_0)(G_\varepsilon^{(nl)}(\zeta_1,\ldots,\zeta_N))=0$, then the function family $G_\varepsilon^{(nl)}(\zeta_1,\ldots,\zeta_N)$ satisfies the annihilation pole condition (2.9).

Proof. In this proof, for $Y \in V^{\otimes N}$ we define $Y^{[\varepsilon_1\varepsilon_2]} \in V^{\otimes N-2}$ by

$$Y = \sum_{\varepsilon_1\varepsilon_2} Y^{[\varepsilon_1\varepsilon_2]} \otimes v_{\varepsilon_1} \otimes v_{\varepsilon_2}$$

and call it the $[\varepsilon_1\varepsilon_2]$ component of Y.

Let $A^{(nl)}_{\varepsilon,\sigma}(\zeta'|\zeta_{N-1})$ and $B^{(nl)}_{\varepsilon,\sigma}(\zeta'|\zeta_{N-1})$ denote the LHS and RHS of the annihilation pole condition (2.9) respectively, and set $K^{(nl)}_{\varepsilon,\sigma}(\zeta'|\zeta_{N-1}) = A^{(nl)}_{\varepsilon,\sigma}(\zeta'|\zeta_{N-1}) - B^{(nl)}_{\varepsilon,\sigma}(\zeta'|\zeta_{N-1})$. The aim of this proof is to show that $K^{(nl)}_{\varepsilon,\sigma}$ vanishes under the four assumptions given above. From the Yang–Baxter equation (2.3) and assumptions (i) and (iii) we obtain

$$K^{(nl)}_{\varepsilon,\sigma}(\zeta'|\zeta_{N-1})^{[++]} = 0. \tag{4.1}$$

The intertwining property of the S-matrix implies

$$\begin{aligned} &S_{N-1|1,\ldots,N-2}(\zeta_{N-1}|\zeta')(\pi_{\zeta'} \otimes \pi_{\zeta_{N-1}}) \circ \Delta'(y) \\ &\quad = (\pi_{\zeta'} \otimes \pi_{\zeta_{N-1}}) \circ \Delta(y) S_{N-1|1,\ldots,N-2}(\zeta_{N-1}|\zeta') \qquad y \in U. \end{aligned}$$

Therefore, noting $G^{(n-1\,l-1)}_{\varepsilon} \in V^{(n-1\,l-l)}$ we find from assumption (iv) that

$$\pi_{(\zeta',\zeta_{N-1},-\sigma\zeta_{N-1}\tau)}(f_0) K^{(nl)}_{\varepsilon,\sigma}(\zeta'|\zeta_{N-1}) = 0. \tag{4.2}$$

Equations (4.1) and (4.2) imply that $K^{(nl)}_{\varepsilon,\sigma}$ has the form

$$K^{(nl)}_{\varepsilon,\sigma}(\zeta'|\zeta_{N-1}) = K'^{(nl)}_{\varepsilon,\sigma}(\zeta'|\zeta_{N-1}) \otimes u_{\sigma}.$$

We shall now show $K'^{(nl)}_{\varepsilon,\sigma} = 0$. Assumptions (i) and (ii) give the q-KZ equation

$$P_{N-1\,N} G^{(nl)}_{\varepsilon}(\zeta', \zeta_{N-1}, \zeta_N) = r^{(l-n)}_{\varepsilon}(\zeta_N q^2) D^{(l-n)}_{N-1} S_{1,\ldots,N-2|N-1}(\zeta'|\zeta_N q^2) G^{(nl)}_{\varepsilon}(\zeta', \zeta_N q^2, \zeta_{N-1}). \tag{4.3}$$

Taking the residue at $\zeta_N = -\sigma\zeta_{N-1}\tau$ we obtain

$$P_{N-1\,N} A^{(nl)}_{\varepsilon,\sigma}(\zeta'| - \sigma\zeta_{N-1}\tau) = -r^{(l-n)}_{\varepsilon}(\zeta_{N-1}) D^{(l-n)}_{N-1} S_{1,\ldots,N-2|N-1}(\zeta'|\zeta_{N-1}) A^{(nl)}_{\varepsilon,\sigma}(\zeta'|\zeta_{N-1}).$$

From the unitary and crossing symmetry of the S-matrix (2.5), (2.6), and the condition for $r^{(k)}(\zeta)$ (2.10), it follows that $B^{(nl)}_{\varepsilon,\sigma}(\zeta'|\zeta_{N-1})$ also satisfies the above equation. From these we find

$$\begin{aligned} &\sigma K'^{(nl)}_{\varepsilon,\sigma}(\zeta'| - \sigma\zeta_{N-1}\tau) \otimes u_{\sigma} \\ &\quad = -r^{(l-n)}_{\varepsilon}(\zeta_{N-1}) D^{(l-n)}_{N-1} S_{1,\ldots,N-2|N-1}(\zeta'|\zeta_{N-1}) K'^{(nl)}_{\varepsilon,\sigma}(\zeta'|\zeta_{N-1}) \otimes u_{\sigma}. \end{aligned}$$

By considering the difference of the $[-+]$ component and $\sigma\times$ the $[+-]$ component of the above equation, we obtain

$$M^{(nl)}(\zeta'|\zeta_{N-1}) K'^{(nl)}_{\varepsilon,\sigma}(\zeta'|\zeta_{N-1}) = 0.$$

Here

$$\begin{aligned} &M^{(nl)}(\zeta'|\zeta_{N-1}) = \mathrm{tr}_{N-1}\left(\bar{D}^{(l-n)}_{N-1} R_{1,\ldots,N-2|N-1}(\zeta'|\zeta_{N-1})\right) \in \mathrm{End}(V^{\otimes N-2}) \\ &\bar{D}^{(k)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} D^{(k)} \end{aligned}$$

and tr_{N-1} signifies the trace on the $(N-1)$th space. Since the matrix $M^{(nl)}$ is invertible for generic ζ_j (for example, consider the special case $\zeta_1 = \ldots = \zeta_{N-1}$), we obtain $K'^{(nl)}_{\varepsilon,\sigma}(\zeta'|\zeta_{N-1}) = 0$. □

We have already shown that the function family $G_\varepsilon^{(nl)}(\zeta)$ defined in section 2 satisfies the assumptions (i)–(iii) of proposition 4.1. Therefore, theorem 2.1 is proved if we prove the following lemma.

Lemma 4.2.

$$(\pi_{(\zeta_1,\ldots,\zeta_N)}(f_0))\langle\Delta^{(nl)}\rangle(x_1,\ldots,x_m|\zeta_1,\ldots,\zeta_N)=0. \tag{4.4}$$

Proof. This lemma is proved in a similar manner to lemma 3 in [4, ch 7]. Set

$$\begin{aligned}P^{(n-1\,l+1)}&(x_1,\ldots,x_m|z_1,\ldots,z_{n-1}|z_n,\ldots,z_N)\\&=(-\tau)^l\prod_{j=1}^{n-1}\zeta_j^{-1}\prod_{i=n}^{N}\prod_{j=1}^{n-1}(z_i-z_j\tau^2)\left(\pi_{(\zeta_1,\ldots,\zeta_N)}(f_0)\langle\Delta^{(nl)}\rangle\right.\\&\quad\times\left.(x_1,\ldots,x_m|\zeta_1,\ldots,\zeta_N)\right)^{-\cdots-+\cdots+}.\end{aligned}$$

Then, due to the R symmetry we obtain

$$\begin{aligned}P^{(n-1\,l+1)}&(x_1,\ldots,x_m|z_1,\ldots,z_{n-1}|z_n,\ldots,z_N)\\&=\sum_{k=n}^{N}\frac{\prod_{j=1}^{n-1}(z_k-z_j\tau^2)}{\prod_{\substack{i=n\\i\neq k}}^{N}(z_k-z_i)}\Delta^{(nl)}(x_1,\ldots,x_m|z_1,\ldots,z_{n-1},z_k|z_n,\overset{k}{\hat{\ldots}},z_N).\end{aligned}$$

From the recurrence relation of $\Delta^{(nl)}$ [1] we can see that $P^{(n-1\,l+1)}$ satisfies

$$\begin{aligned}P^{(n-1\,l+1)}&(x_1,\ldots,x_m|z',a|z'',a\tau^2)\\&=\prod_{\mu=1}^{m}(x_\mu-a\tau)\sum_{\nu=1}^{m}(-1)^{m+\nu}h^{(N-2)}(x_\nu|z',z'')P^{(n-2\,l)}(x_1,\overset{\nu}{\hat{\ldots}},x_m|z'|z'')\end{aligned}$$

where $z'=(z_1,\ldots,z_{n-2})$ and $z''=(z_n,\ldots,z_{N-1})$. We further find from the properties of $\Delta^{(nl)}$ as a polynomial [1] that $P^{(n-1\,l+1)}$ is a homogeneous polynomial of degree $\binom{m}{2}+(n-1)l-1$, antisymmetric with respect to the x_μ's and symmetric with respect to $\{z_1,\ldots,z_{n-1}\}$ and $\{z_n,\ldots,z_N\}$. Moreover, from the power counting we obtain $P^{(0\,l+1)}=0$. From these properties we can show $P^{(n-1\,l+1)}=0$. □

5. Discussion

In this section we discuss the function family $G_\varepsilon^{(nl)}(\zeta)$ satisfying the three axioms (2.7)–(2.9) in the context of the XXZ model in the ferroelectric regime ($-1<q<0$). In [6,7] this model was solved by utilizing the representation theory of $U_q(\widehat{\mathfrak{sl}_2})$. First, following their notation, we summarize the necessary results for this model. (See [6,7] for further details.) Let $V(\Lambda_i)$, $i=0,1$, be the level 1 highest weight module of $U_q(\widehat{\mathfrak{sl}_2})$ and set $\mathcal{H}=V(\Lambda_0)\oplus V(\Lambda_1)$ and $\mathcal{F}^{(ij)}=V(\Lambda_i)\otimes V(\Lambda_j)^{*b}$. Here the superscript $*b$ signifies the dual module regarded as a left module by some anti-automorphism b. Then, the space $\mathcal{F}$ on which the XXZ Hamiltonian acts is identified with

$$\mathcal{F}=\mathcal{H}\otimes\mathcal{H}^{*b}=\bigoplus_{i,j=0,1}\mathcal{F}^{(ij)}.$$

In this space $\mathcal{F}$, there exist two ground states of the Hamiltonian which belong to $\mathcal{F}^{(00)}$ and $\mathcal{F}^{(11)}$. We denote them and their dual vectors by $|\mathrm{vac}\rangle_{(i)}$ and ${}_{(i)}\langle\mathrm{vac}|$, $i=0,1$, respectively. The creation and annihilation operators $\varphi_\varepsilon^*(\zeta),\varphi^\varepsilon(\zeta)\in\oplus_{ij}\mathrm{Hom}(\mathcal{F}^{(ij)},\mathcal{F}^{(1-i\,j)})$, $\varepsilon=$

$\pm, |\zeta| = 1$, that diagonalize the Hamiltonian can be constructed in terms of the vertex operators of the algebra. They have the property that

$$\varphi^{\varepsilon}(\zeta)|\text{vac}\rangle_{(i)} = 0 \qquad {}_{(i)}\langle\text{vac}|\varphi^{*}_{\varepsilon}(\zeta) = 0$$

and the whole space $\mathcal{F}$ and its dual space $\mathcal{F}^{*}$ are spanned by the vectors

$$\varphi^{*}_{\varepsilon_1}(\zeta_1)\cdots\varphi^{*}_{\varepsilon_N}(\zeta_N)|\text{vac}\rangle_{(i)} \text{ and } {}_{(i)}\langle\text{vac}|\varphi^{\varepsilon_1}(\zeta_1)\cdots\varphi^{\varepsilon_N}(\zeta_N) \qquad (i = 0, 1,\ N = 0, 1, \ldots)$$

respectively. Further, on $\mathcal{F}^{(ij)}$ these operators satisfy

$$\varphi^{*}_{\varepsilon}(-\zeta) = (-1)^{(\varepsilon-(-1)^{i})/2}\varphi^{*}_{\varepsilon}(\zeta) \qquad \varphi^{\varepsilon}(-\zeta) = (-1)^{(\varepsilon+(-1)^{i})/2}\varphi^{\varepsilon}(\zeta) \qquad (i, j = 0, 1) \tag{5.1}$$

and the commutation relations

$$\begin{aligned}
&P_{12}\varphi^{V}(\zeta_1)\varphi^{V}(\zeta_2) = S_{12}(\zeta_1/\zeta_2)\varphi^{V}(\zeta_2)\varphi^{V}(\zeta_1)\\
&P_{12}\varphi^{*V}(\zeta_1)\varphi^{*V}(\zeta_2) = S_{12}(\zeta_1/\zeta_2)\varphi^{*V}(\zeta_2)\varphi^{*V}(\zeta_1)\\
&P_{12}\varphi^{V}(\zeta_1)\varphi^{*V}(\zeta_2) = S_{12}(-\zeta_1/q\zeta_2)\varphi^{*V}(\zeta_2)\varphi^{V}(\zeta_1) + \frac{1}{2}\sum_{\sigma=\pm}\sigma\delta(\sigma\zeta_1/\zeta_2)u_{\sigma}\otimes(P_0+\sigma P_1).
\end{aligned}$$

In the above equations,

$$\varphi^{*V}(\zeta) = \sum_{\varepsilon} v_{\varepsilon}\otimes\varphi^{*}_{-\varepsilon}(\zeta) \qquad \varphi^{V}(\zeta) = \sum_{\varepsilon} v_{\varepsilon}\otimes\varphi^{\varepsilon}(\zeta) \in \bigoplus_{ij}\text{Hom}(\mathcal{F}^{(ij)}, V\otimes\mathcal{F}^{(1-i\,j)})$$

P_i is the projection operator to the subspace $\oplus_{j=0,1}\mathcal{F}^{(ij)}$ and $\delta(\zeta) = \sum_{m\in\mathbb{Z}}\zeta^{m}$. Note that we consider the creation and annihilation operators in the principal picture. (See, for example, equations (2.2), (4.8) and (4.9) in [11] for the above properties of these operators in the principal picture.)

Now we shall discuss the form factors of this model. For simplicity, we shall consider an operator $\mathcal{O} \in \text{End}(\mathcal{F})$ that has the form $\text{id}\otimes O$, $O \in \text{End}(\mathcal{H}^{*})$, and satisfies

$$(\text{id}\otimes\bar{\Psi}^{V}(\zeta))\mathcal{O} = r(-\zeta q)D\mathcal{O}(\text{id}\otimes\bar{\Psi}^{V}(\zeta)). \tag{5.2}$$

Here, $r(\zeta)$ is a scalar function, D is a diagonal matrix acting on the space V and $\bar{\Psi}^{V}(\zeta) : \mathcal{H}^{*b} \to \mathcal{H}^{*b}\otimes V_{\zeta}$ is the type II vertex operator in the terminology of [6, 7]. We denoted the vector representation defined in (2.1) by V_{ζ} . For such $\mathcal{O}$ we introduce the form factor by

$$G^{(N)}_{\varepsilon}(\zeta_1,\ldots,\zeta_N) = \sum_{i=0,1}\varepsilon^{i}\times{}_{(i)}\langle\text{vac}|\mathcal{O}\varphi^{*V}(\zeta_N)\cdots\varphi^{*V}(\zeta_1)|\text{vac}\rangle_{(i+N)} \qquad (\varepsilon = \pm 1).$$

The commutation relations among the creation operators imply the first axiom. From (5.1) we have

$$G^{(N)}_{\varepsilon}(\zeta_1,\ldots,-\zeta_j,\ldots,\zeta_N)^{-\cdots\cdots+\cdots+} = (-1)^{N-j+H(n\geqslant j)}G^{(N)}_{-\varepsilon}(\zeta_1,\ldots,\zeta_j,\ldots,\zeta_N)^{-\cdots\cdots+\cdots+} \tag{5.3}$$

where H is the step function and n is the number of the superscript $(-)$ of $G^{(N)}_{\varepsilon}$. Note that for $G^{(nl)}_{\varepsilon}(\zeta)$ defined in section 2, condition (5.3) is equivalent to

$$\vartheta^{(nl)}_{\varepsilon}(x_1,\ldots,x_m|\zeta_1,\ldots,-\zeta_j,\ldots,\zeta_N) = \vartheta^{(nl)}_{-\varepsilon}(x_1,\ldots,x_m|\zeta_1,\ldots,\zeta_j,\ldots,\zeta_N). \tag{5.4}$$

Let $\zeta^{\pm}$ signify $(1\pm\eta)\zeta$, $0<\eta\ll 1$, for ζ, and ζ'' be the abbreviation $(\zeta_1,\ldots,\zeta_{N-1})$. Set $r_{\varepsilon}(\zeta) = \varepsilon r(\zeta)$ as before and set

$$G^{(N-2)}_{\varepsilon,\sigma,k}(\zeta_1,\ldots,\zeta_{N-1}) = P_{k,k+1}\cdots P_{N-2,N-1}G^{(N-2)}_{\sigma\varepsilon}(\zeta_1,\overset{k}{\hat{\ldots}},\zeta_{N-1})\otimes u_{\sigma}.$$

Then, as in [11], thanks to condition (5.2) the following matrix element of $\mathcal{O}$ can be expressed in terms of the form factors in two different ways:

$$\sum_{i=0,1} \varepsilon^i \times {}_{(i)}\langle \mathrm{vac}|\varphi^V(\zeta_N)\mathcal{O}\varphi^{*V}(\zeta_{N-1})\cdots\varphi^{*V}(\zeta_1)|\mathrm{vac}\rangle_{(i+N)}$$
$$= G_\varepsilon^{(N)}(\zeta'', \zeta_N^-/(-q)) + \tfrac{1}{2}\sum_{\substack{1\leqslant k\leqslant N-1\\ \sigma=\pm}} \delta(\sigma\zeta_k/\zeta_N)S_{k+1,\ldots,N-1|k}$$
$$\times(\zeta_{k+1},\ldots,\zeta_{N-1}|\zeta_k)G_{\varepsilon,\sigma,k}^{(N-2)}(\zeta'')$$
$$= r_\varepsilon(-\zeta_N q)D_N\Big(P_{N-1,N}\cdots P_{1,2}G_\varepsilon^{(N)}(-q\zeta_N^+, \zeta'')$$
$$+\tfrac{1}{2}\sum_{\substack{1\leqslant k\leqslant N-1\\ \sigma=\pm}} \sigma^{N+1}\delta(\sigma\zeta_k/\zeta_N)S_{k|1,\ldots,k-1}(\zeta_k|\zeta_1,\ldots,\zeta_{k-1})G_{\varepsilon,\sigma,k}^{(N-2)}(\zeta'')\Big). \qquad (5.5)$$

The last equality of (5.5) implies that $G_\varepsilon^{(N)}(\zeta)$ also satisfies the second and third axioms. Conversely, suppose that we are given $G_\varepsilon^{(nl)}(\zeta)$ satisfying the three axioms and (5.3); we can then define an operator $\mathcal{O} \in \mathrm{End}(\mathcal{F})$ by giving the matrix elements in terms of $G_\varepsilon^{(nl)}(\zeta)$ in two different ways as done in the original work [4]. The second and third axioms ensures the equivalence of the two expressions. (Equation (5.5) shows one simple example.) These two expressions enable us to calculate the commutation relations of the thus defined operators, though knowledge of the poles, other than annihilation poles, of the $G_\varepsilon^{(nl)}(\zeta)$ is necessary. The classification of all solutions to the conditions (2.15)–(2.17), and (5.4) and the identification of them with $\mathcal{O} \in \mathrm{End}(\mathcal{F})$ are still open questions.

Finally, we shall give an example of $\vartheta_\varepsilon^{(nl)}$ in the case

$$r_\varepsilon^{(nl)}(\zeta) = \varepsilon\prod_{k=1}^{N-2m} t(\zeta/\xi_k) \qquad t(\zeta) = \zeta^{-1}\frac{\theta(qz|q^4)}{\theta(qz^{-1}|q^4)}$$

where $m = n-1$ for $n = l$ and $m = n$ for $n < l$ as before, and $\xi_k \in \mathbb{C}$. In this case one solution to the conditions (2.15)–(2.17) and (5.4) is given by

$$\frac{\vartheta_\varepsilon^{(nl)}(x_1,\ldots,x_m|\zeta_1,\ldots,\zeta_N)}{\prod_{j=1}^N \zeta_j^{N-2m}\prod_{1\leqslant\mu<\nu\leqslant m}\left(x_\mu\theta(x_\nu/x_\mu|q^2)\right)}$$
$$= \mathrm{constant}\,\varepsilon^N\theta(-\varepsilon u|q^2)\prod_{\substack{1\leqslant k\leqslant N-2m\\ 1\leqslant\mu\leqslant m}}\theta(\xi_k^2/x_\mu|q^4)\prod_{\substack{1\leqslant j\leqslant N\\ 1\leqslant k\leqslant N-2m}} f(z_j/\xi_k^2) \qquad (5.6)$$

where

$$u = (-1)^{N-m}\tau^{N/2-3m}\frac{\prod_{\mu=1}^m x_\mu\prod_{k=1}^{N-2m}\xi_k}{\prod_{j=1}^N\zeta_j} \qquad f(z) = \frac{(q^5z;q^4,q^4)_\infty(q^5/z;q^4,q^4)_\infty}{(q^3z;q^4,q^4)_\infty(q^3/z;q^4,q^4)_\infty}.$$

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